Kurt Godel (1961) - The Modern Development Of The Foundations Of Mathematics In The Light Of Philosophy
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Source: Kurt Godel, Collected Works, Volume III (1961) publ. Oxford University Press, 1981.
The Complete lecture reproduced here.

I would like to attempt here to describe, in terms of philosophical concepts, the development of foundational research in mathematics since around the turn of the century, and to fit it into a general schema of possible philosophical world-views [Weltanschauungen]. For this, it is necessary first of all to become clear about the schema itself. I believe that the most fruitful principle for gaining an overall view of the possible world-views will be to divide them up according to the degree and the manner of their affinity to or, respectively, turning away from metaphysics (or religion). In this way we immediately obtain a division into two groups: scepticism, materialism and positivism stand on one side, spiritualism, idealism and theology on the other. We also at once see degrees of difference in this sequence, in that scepticism stands even farther away from theology than does materialism, while on the other hand idealism, e.g., in its pantheistic form, is a weakened form of theology in the proper sense.

The schema also proves fruitful, however, for the analysis of philosophical doctrines admissible in special contexts, in that one either arranges them in this manner or, in mixed cases, seeks out their materialistic and spiritualistic elements. Thus one would, for example, say that apriorism belongs in principle on the right and empiricism on the left side. On the other hand, however, there are also such mixed forms as an empiristically grounded theology. Furthermore one sees also that optimism belongs in principle toward the right and pessimism toward the left. For scepticism is certainly a pessimism with regard to knowledge. Moreover, materialism is inclined to regard the world as an unordered and therefore meaningless heap of atoms. In addition, death appears to it to be final and complete annihilation, while, on the other hand, theology and idealism see sense, purpose and reason in everything. On the other hand, Schopenhauer's pessimism is a mixed form, namely a pessimistic idealism. Another example of a theory evidently on the right is that of an objective right and objective aesthetic values, whereas the interpretation of ethics and aesthetics on the basis of custom, upbringing, etc., belongs toward the left.

Now it is a familiar fact, even a platitude, that the development of philosophy since the Renaissance has by and large gone from right to left - not in a straight line, but with reverses, yet still, on the whole. Particularly in physics, this development has reached a peak in our own time, in that, to a large extent, the possibility of knowledge of the objectivisable states of affairs is denied, and it is asserted that we must be content to predict results of observations. This is really the end of all theoretical science in the usual sense (although this predicting can be completely sufficient for practical purposes such as making television sets or atom bombs).

It would truly be a miracle if this (I would like to say rabid) development had not also begun to make itself felt in the conception of mathematics. Actually, mathematics, by its nature as an a priori science, always has, in and of itself, an inclination toward the right, and, for this reason, has long withstood the spirit of the time [Zeitgeist] that has ruled since the Renaissance; i.e., the empiricist theory of mathematics, such as the one set forth by Mill, did not find much support. Indeed, mathematics has evolved into ever higher abstractions, away from matter and to ever greater clarity in its foundations (e.g., by giving an exact foundation of the infinitesimal calculus and the complex numbers) - thus, away from scepticism.

Finally, however, around the turn of the century, its hour struck: in particular, it was the antinomies of set theory, contradictions that allegedly appeared within mathematics, whose significance was exaggerated by sceptics and empiricists and which were employed as a pretext for the leftward upheaval. I say "allegedly" and "exaggerated" because, in the first place, these contradictions did not appear within mathematics but near its outermost boundary toward philosophy, and secondly, they have been resolved in a manner that is completely satisfactory and, for everyone who understands the theory, nearly obvious. Such arguments are, however, of no use against the spirit of the time, and so the result was that many or most mathematicians denied that mathematics, as it had developed previously, represents a system of truths; rather, they acknowledged this only for a part of mathematics (larger or smaller, according to their temperament) and retained the rest at best in a hypothetical sense namely, one in which the theory properly asserts only that from certain assumptions (not themselves to be justified), we can justifiably draw certain conclusions. They thereby flattered themselves that everything essential had really been retained. Since, after all, what interests the mathematician, in addition to drawing consequences from these assumptions, is what can be carried out. In truth, however, mathematics becomes in this way an empirical science. For if I somehow prove from the arbitrarily postulated axioms that every natural number is the sum of four squares, it does not at all follow with certainty that I will never find a counter-example to this theorem, for my axioms could after all be inconsistent, and I can at most say that it follows with a certain probability, because in spite of many deductions no contradiction has so far been discovered. In addition, through this hypothetical conception of mathematics, many questions lose the form "Does the proposition A hold or not?" For, from assumptions construed as completely arbitrary, I can of course not expect that they have the peculiar property of implying, in every case, exactly either A or ~A.

Although these nihilistic consequences are very well in accord with the spirit of the time, here a reaction set in obviously not on the part of philosophy, but rather on that of mathematics, which, by its nature, as I have already said, is very recalcitrant in the face of the Zeitgeist. And thus came into being that curious hermaphroditic thing that Hilbert's formalism represents, which sought to do justice both to the spirit of the time and to the nature of mathematics. It consists in the following: on the one hand, in conformity with the ideas prevailing in today's philosophy, it is acknowledged that the truth of the axioms from which mathematics starts out cannot be justified or recognised in any way, and therefore the drawing of consequences from them has meaning only in a hypothetical sense, whereby this drawing of consequences itself (in order to satisfy even further the spirit of the time) is construed as a mere game with symbols according to certain rules, likewise not supported by insight.

But, on the other hand, one clung to the belief, corresponding to the earlier "rightward" philosophy of mathematics and to the mathematician's instinct, that a proof for the correctness of such a proposition as the representability of every number as a sum of four squares must provide a secure grounding for that proposition - and furthermore, also that every precisely formulated yes-or-no question in mathematics must have a clear-cut answer. I.e., one thus aims to prove, for inherently unfounded rules of the game with symbols, as a property that attaches to them so to speak by accident, that of two sentences A and ~A, exactly one can always be derived. That not both can be derived constitutes consistency, and that one can always actually be derived means that the mathematical question expressed by A can be unambiguously answered. Of course, if one wishes to justify these two assertions with mathematical certainty, a certain part of mathematics must be acknowledged as true in the sense of the old rightward philosophy. But that is a part that is much less opposed to the spirit of the time than the high abstractions of set theory. For it refers only to concrete and finite objects in space, namely the combinations of symbols.

What I have said so far are really only obvious things, which I wanted to recall merely because they are important for what follows. But the next step in the development is now this: it turns out that it is impossible to rescue the old rightward aspects of mathematics in such a manner as to be more or less in accord with the spirit of the time. Even if we restrict ourselves to the theory of natural numbers, it is impossible to find a system of axioms and formal rules from which, for every number-theoretic proposition A, either A or ~A would always be derivable. And furthermore, for reasonably comprehensive axioms of mathematics, it is impossible to carry out a proof of consistency merely by reflecting on the concrete combinations of symbols, without introducing more abstract elements. The Hilbertian combination of materialism and aspects of classical mathematics thus proves to be impossible.

Hence, only two possibilities remain open. One must either give up the old rightward aspects of mathematics or attempt to uphold them in contradiction to the spirit of the time. Obviously the first course is the only one that suits our time and is therefore also the one usually adopted. One should, however, keep in mind that this is a purely negative attitude. One simply gives up aspects whose fulfilment would in any case be very desirable and which have much to recommend themselves: namely, on the one hand, to safeguard for mathematics the certainty of its knowledge, and on the other, to uphold the belief that for clear questions posed by reason, reason can also find clear answers. And as should be noted, one gives up these aspects not because the mathematical results achieved compel one to do so but because that is the only possible way, despite these results, to remain in agreement with the prevailing philosophy.

Now one can of course by no means close one's eyes to the great advances which our time exhibits in many respects, and one can with a certain justice assert that these advances are due just to this leftward spirit in philosophy and world-view. But, on the other hand, if one considers the matter in proper historical perspective, one must say that the fruitfulness of materialism is based in part only on the excesses and the wrong direction of the preceding rightward philosophy. As far as the rightness and wrongness, or, respectively, truth and falsity, of these two directions is concerned, the correct attitude appears to me to be that the truth lies in the middle or consists of a combination of the two conceptions.

Now, in the case of mathematics, Hilbert had of course attempted just such a combination, but one obviously too primitive and tending too strongly in one direction. In any case there is no reason to trust blindly in the spirit of the time, and it is therefore undoubtedly worth the effort at least once to try the other of the alternatives mentioned above, which the results cited leave open - in the hope of obtaining in this way a workable combination. Obviously, this means that the certainty of mathematics is to be secured not by proving certain properties by a projection onto material systems - namely, the manipulation of physical symbols but rather by cultivating (deepening) knowledge of the abstract concepts themselves which lead to the setting up of these mechanical systems, and further by seeking, according to the same procedures, to gain insights into the solvability, and the actual methods for the solution, of all meaningful mathematical problems.

In what manner, however, is it possible to extend our knowledge of these abstract concepts, i.e., to make these concepts themselves precise and to gain comprehensive and secure insight into the fundamental relations that subsist among them, i.e., into the axioms that hold for them? Obviously not, or in any case not exclusively, by trying to give explicit definitions for concepts and proofs for axioms, since for that one obviously needs other undefinable abstract concepts and axioms holding for them. Otherwise one would have nothing from which one could define or prove. The procedure must thus consist, at least to a large extent, in a clarification of meaning that does not consist in giving definitions.

Now in fact, there exists today the beginning of a science which claims to possess a systematic method for such a clarification of meaning, and that is the phenomenology founded by Husserl. Here clarification of meaning consists in focusing more sharply on the concepts concerned by directing our attention in a certain way, namely, onto our own acts in the use of these concepts, onto our powers in carrying out our acts, etc. But one must keep clearly in mind that this phenomenology is not a science in the same sense as the other sciences. Rather it is or in any case should be a procedure or technique that should produce in us a new state of consciousness in which we describe in detail the basic concepts we use in our thought, or grasp other basic concepts hitherto unknown to us. I believe there is no reason at all to reject such a procedure at the outset as hopeless. Empiricists, of course, have the least reason of all to do so, for that would mean that their empiricism is, in truth, an apriorism with its sign reversed.

But not only is there no objective reason for the rejection of phenomenology, but on the contrary one can present reasons in its favour. If one considers the development of a child, one notices that it proceeds in two directions: it consists on the one hand in experimenting with the objects of the external world and with its own sensory and motor organs, on the other hand in coming to a better and better understanding of language, and that means - as soon - as the child is beyond the most primitive designating of objects - of the basic concepts on which it rests. With respect to the development in this second direction, one can justifiably say that the child passes through states of consciousness of various heights, e.g., one can say that a higher state of consciousness is attained when the child first learns the use of words, and similarly at the moment when for the first time it understands a logical inference.

Now one may view the whole development of empirical science as a systematic and conscious extension of what the child does when it develops in the first direction. The success of this procedure is indeed astonishing and far greater than one would expect a priori: after all, it leads to the entire technological development of recent times. That makes it thus seem quite possible that a systematic and conscious advance in the second direction will also far exceed the expectations one may have a priori.

In fact, one has examples where, even without the application of a systematic and conscious procedure, but entirely by itself, a considerable further development takes place in the second direction, one that transcends "common sense". Namely, it turns out that in the systematic establishment of the axioms of mathematics, new axioms, which do not follow by formal logic from those previously established, again and again become evident. It is not at all excluded by the negative results mentioned earlier that nevertheless every clearly posed mathematical yes-or-no question is solvable in this way. For it is just this becoming evident of more and more new axioms on the basis of the meaning of the primitive notions that a machine cannot imitate.

I would like to point out that this intuitive grasping of ever newer axioms that are logically independent from the earlier ones, which is necessary for the solvability of all problems even within a very limited domain, agrees in principle with the Kantian conception of mathematics. The relevant utterances by Kant are, it is true, incorrect if taken literally, since Kant asserts that in the derivation of geometrical theorems we always need new geometrical intuitions, and that therefore a purely logical derivation from a finite number of axioms is impossible. That is demonstrably false. However, if in this proposition we replace the term "geometrical" - by "mathematical" or "set-theoretical", then it becomes a demonstrably true proposition. I believe it to be a general feature of many of Kant's assertions that literally understood they are false but in a broader sense contain deep truths. In particular, the whole phenomenological method, as I sketched it above, goes back in its central idea to Kant, and what Husserl did was merely that he first formulated it more precisely, made it fully conscious and actually carried it out for

particular domains. Indeed, just from the terminology used by Husserl, one sees how positively he himself values his relation to Kant.

I believe that precisely because in the last analysis the Kantian philosophy rests on the idea of phenomenology, albeit in a not entirely clear way, and has just thereby introduced into our thought something completely new, and indeed characteristic of every genuine philosophy - it is precisely on that, I believe, that the enormous influence which Kant has exercised over the entire subsequent development of philosophy rests. Indeed, there is hardly any later direction that is not somehow related to Kant's ideas. On the other hand, however, just because of the lack of clarity and the literal incorrectness of many of Kant's formulations, quite divergent directions have developed out of Kant's thought - none of which, however, really did justice to the core of Kant's thought. This requirement seems to me to be met for the first time by phenomenology, which, entirely as intended by Kant, avoids both the death-defying leaps of idealism into a new metaphysics as well as the positivistic rejection of all metaphysics. But now, if the misunderstood Kant has already led to so much that is interesting in philosophy, and also indirectly in science, how much more can we expect it from Kant understood correctly?

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